A NOTE ON 2-DISTANT NONCROSSING PARTITIONS AND WEIGHTED MOTZKIN PATHS

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Abstract. We prove a conjecture of Drake and Kim: the number of 2-distant noncrossing partitions of $\{1, 2, \ldots, n\}$ is equal to the sum of weights of Motzkin paths of length n, where the weight of a Motzkin path is a product of certain fractions involving Fibonacci numbers. We provide two proofs of their conjecture: one uses continued fractions and the other is combinatorial.

1. Introduction

A *Motzkin path* of length n is a lattice path from $(0, 0)$ to $(n, 0)$ consisting of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (1, 0)$ that never goes below the x-axis. The *height* of a step in a Motzkin path is the y coordinate of the ending point.

Given two sequences $b = (b_0, b_1, \ldots)$ and $\lambda = (\lambda_0, \lambda_1, \ldots)$, the *weight* of a Motzkin path with respect to (b, λ) is the product of b_i and λ_i for each horizontal step and down step of height i respectively, see Figure 1. Let $\mathrm{Mot}_n(b, \lambda)$ denote the sum of weights of Motzkin paths of length n with respect to (b, λ). This sum is closely related to orthogonal polynomials; see [5, 6].

Drake and Kim [1] defined the set $\mathrm{NC}_k(n)$ of k-distant noncrossing partitions of $[n] = \{1, 2, \ldots, n\}$. For $k \geq 0$, a *k-distant noncrossing partition* is a set partition of $[n]$ without two arcs (a, c) and (b, d) satisfying $a < b \leq c < d$ and $c - b \geq k$, where an *arc* is a pair (i, j) of integers contained in the same block which does not contain any integer between them. For example, $\pi = \{\{1, 5, 7\}, \{2, 3, 6\}, \{4\}\}$ is a 3-distant noncrossing partition but not a 2-distant noncrossing partition because π has two arcs $(1, 5)$ and $(3, 6)$ with $5 - 3 \geq 2$. Note that the 1-distant noncrossing partitions are the ordinary noncrossing partitions, which implies that $\# \mathrm{NC}_1(n)$ is equal to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. It is not difficult to see that $\mathrm{NC}_0(n)$ is in bijection

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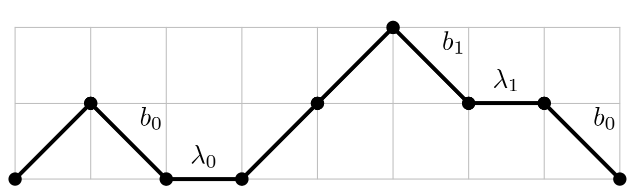

Figure 1. A Motzkin path and the weights of its steps with respect to (b, λ).

with the set of Motzkin paths of length n. In the same paper, they proved that

$$\sum_{n\geq 0} \#\,\mathrm{NC}_2(n)x^n = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{1-5x}{1-x}}. \tag{1}$$

The number $\#\,\mathrm{NC}_2(n)$ also counts many combinatorial objects: Schröder paths with no peaks at even levels, etc; see [2, 4, 7].

There are simple expressions of $\#\,\mathrm{NC}_k(n)$ using Motzkin paths for $k = 0, 1, 3$:

$$\begin{aligned}
\#\,\mathrm{NC}_0(n) &= \mathrm{Mot}_n((1,1,\dots),(1,1,\dots)),\\
\#\,\mathrm{NC}_1(n) &= \mathrm{Mot}_n((1,2,2,\dots),(1,1,\dots)),\\
\#\,\mathrm{NC}_3(n) &= \mathrm{Mot}_n((1,2,3,3,\dots),(1,2,2,\dots)),
\end{aligned}$$

where the second equation is well known and the third one was first conjectured by Drake and Kim [1] and proved by Kim [3]. The main purpose of this paper is to prove the following theorem which was also conjectured by Drake and Kim [1].

Theorem 1.1. *Let $b = (b_0, b_1, \dots)$ and $\lambda = (\lambda_0, \lambda_1, \dots)$ be the sequences with $b_0 = \lambda_0 = 1$ and for $n \geq 1$,*

$$b_n = 3 - \frac{1}{F_{2n-1}F_{2n-3}} \quad \text{and} \quad \lambda_n = 1 + \frac{1}{F_{2n-1}^2}, \tag{2}$$

where F_m is the Fibonacci number defined by $F_0 = 0, F_1 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for all m (so $F_{-1} = 1$). Then we have

$$\#\,\mathrm{NC}_2(n) = \mathrm{Mot}_n(b, \lambda).$$

Theorem 1.1 is very interesting because it is not even obvious that $\mathrm{Mot}_n(b, \lambda)$ is an integer. In this paper, we give two proofs of Theorem 1.1: one uses continued fractions and the other is combinatorial.

2. Continued fractions

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$, $\beta = (\beta_0, \beta_1, \beta_2, \dots)$, and $c = (c_0, c_1, c_2, \dots)$ be sequences of numbers.

Let $J(x; \alpha_0, \beta_0; \alpha_1, \beta_1; \alpha_2, \beta_2; \dots) = J(x; \alpha, \beta)$ denote the *J-fraction*

$$\cfrac{1}{1 - \alpha_0 x - \cfrac{\beta_0 x^2}{1 - \alpha_1 x - \cfrac{\beta_1 x^2}{1 - \alpha_2 x - \dots}}}$$

and let $S(x; c_0, c_1, c_2, \dots) = S(x; c)$ denote the *S-fraction*

$$\cfrac{1}{1 - \cfrac{c_0 x}{1 - \cfrac{c_1 x}{1 - \dots}}}.$$

A *Dyck path* of length $2n$ is a lattice path from $(0, 0)$ to $(2n, 0)$ consisting of up steps $U = (1, 1)$ and down steps $D = (1, -1)$ that never goes below the x-axis. The *height* of a step in a Dyck path is the y coordinate of the ending point. The *weight* of a Dyck path with respect to c is the product of c_i for each down step of height i, see Figure 2. Let $\mathrm{Dyck}_n(c)$ denote the sum of weights of Dyck paths of length $2n$ with respect to c.

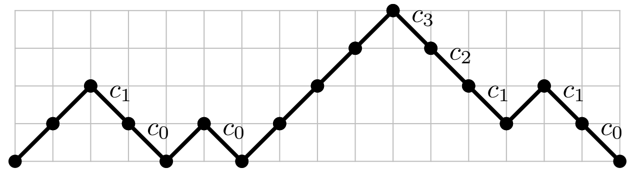

FIGURE 2. A Dyck path and the weights of its steps with respect to c.

It is well known that

$$\sum_{n\geq 0} \mathrm{Mot}_n(\alpha,\beta)x^n = J(x;\alpha,\beta) \quad \text{and} \quad \sum_{n\geq 0} \mathrm{Dyck}_n(c)x^n = S(x;c).$$

The following proposition is easy to see.

Proposition 2.1. *If* $\alpha_n = c_{2n-1} + c_{2n}$ *and* $\beta_n = c_{2n}c_{2n+1}$ *for all* $n \geq 0$*, with* $c_{-1} = 0$*, then* $S(x;c) = J(x;\alpha,\beta)$*.*

One can prove Proposition 2.1 by the following observation: a Motzkin path may be obtained from a Dyck path by taking steps two at a time and changing UU, UD, DU and DD, respectively, to U, H, H and D. For example, the Motzkin path in Figure 1 is obtained from the Dyck path in Figure 2 in this way.

Let $d = (d_0, d_1, d_2, \dots)$ be the sequence with $d_0 = 1$ and for $n \geq 1$,

$$d_{2n-1} = \frac{F_{2n-1}}{F_{2n-3}}, \qquad d_{2n} = \frac{1}{d_{2n-1}}. \tag{3}$$

Recall the two sequences $b = (b_0, b_1, \dots)$ and $\lambda = (\lambda_0, \lambda_1, \dots)$ defined in (2).

Lemma 2.2. *We have the following.*

(1) $b_n = d_{2n-1} + d_{2n}$ *for all* $n \geq 0$*, where* $d_{-1} = 0$*.*
(2) $\lambda_n = d_{2n}d_{2n+1}$ *for all* $n \geq 0$*.*
(3) $1/d_{2n-1} + d_{2n+1} = 3$ *for all* $n \geq 1$*.*

Proof. We will use two cases of the well-known Catalan identity for Fibonacci numbers, $F_m^2 - F_{m+i}F_{m-i} = (-1)^{m-i}F_i^2$.

(1) This is true for $n = 0$. For $n \geq 1$ we have

$$\begin{aligned} d_{2n-1} + d_{2n} &= \frac{F_{2n-1}}{F_{2n-3}} + \frac{F_{2n-3}}{F_{2n-1}} = \frac{F_{2n-1}^2 + F_{2n-3}^2}{F_{2n-1}F_{2n-3}} = \frac{2F_{2n-1}F_{2n-3} + (F_{2n-1} - F_{2n-3})^2}{F_{2n-1}F_{2n-3}} \\ &= 2 + \frac{F_{2n-2}^2}{F_{2n-1}F_{2n-3}} = 3 + \frac{F_{2n-2}^2 - F_{2n-1}F_{2n-3}}{F_{2n-1}F_{2n-3}} = 3 - \frac{1}{F_{2n-1}F_{2n-3}} = b_n. \end{aligned}$$

(2) This is true for $n = 0$. For $n \geq 1$ we have

$$d_{2n}d_{2n+1} = \frac{F_{2n-3}}{F_{2n-1}}\frac{F_{2n+1}}{F_{2n-1}} = \frac{F_{2n-1}^2 + (F_{2n-3}F_{2n+1} - F_{2n-1}^2)}{F_{2n-1}^2} = 1 + \frac{1}{F_{2n-1}^2} = \lambda_n.$$

(3) We have

$$\begin{aligned} \frac{1}{d_{2n-1}} + d_{2n+1} &= \frac{F_{2n-3}}{F_{2n-1}} + \frac{F_{2n+1}}{F_{2n-1}} = \frac{(F_{2n-1} - F_{2n-2}) + (F_{2n} + F_{2n-1})}{F_{2n-1}} \\ &= 2 + \frac{F_{2n} - F_{2n-2}}{F_{2n-1}} = 3. \end{aligned}$$

$\square$

By Proposition 2.1 and Lemma 2.2, we obtain the following.

Corollary 2.3. *For the sequences b, λ and d defined in (2) and (3), we have*

$$\mathrm{Dyck}_n(d) = \mathrm{Mot}_n(b, \lambda).$$

Now we can prove the following S-fraction formula for the generating function (1) for $\#\,\mathrm{NC}_2(n)$.

Theorem 2.4. *We have*

$$\begin{aligned}\frac{3}{2} - \frac{1}{2}\sqrt{\frac{1-5x}{1-x}} &= S(x; 1, 1, 1, 2, \tfrac{1}{2}, \tfrac{5}{2}, \tfrac{2}{5}, \tfrac{13}{5}, \tfrac{5}{13}, \tfrac{34}{13}, \tfrac{13}{34}, \tfrac{89}{34}, \tfrac{34}{89}, \tfrac{233}{89}, \tfrac{89}{233}, \tfrac{610}{233}, \tfrac{233}{610}, \tfrac{1597}{610}, \dots)\\ &= S(x; d_0, d_1, d_2, \dots).\end{aligned}$$

To prove Theorem 2.4, we define R_n for $n \ge -1$ by

$$\begin{aligned}R_{-1} &= \frac{3}{2} - \frac{1}{2}\sqrt{\frac{1-5x}{1-x}},\\ R_{2n+1} &= d_{2n+1} + \frac{1-3x-\sqrt{(1-x)(1-5x)}}{2x}, \quad n \ge 0,\\ R_{2n} &= \frac{d_{2n}}{1-xR_{2n+1}}, \quad n \ge 0.\end{aligned}$$

One can easily check that R_m is a power series in x with constant term d_m (with $d_{-1} = 1$), though this will follow from Lemma 2.5.

Lemma 2.5. *For $m \ge -1$, we have*

$$R_m = \frac{d_m}{1 - xR_{m+1}},$$

where $d_{-1} = 1$.

Proof. By definition, this is true if m is even. Thus it is enough to prove that for $n \ge 0$,

$$R_{2n-1} = \frac{d_{2n-1}}{1-xR_{2n}} = \frac{d_{2n-1}}{1 - \dfrac{xd_{2n}}{1-xR_{2n+1}}},$$

which is equivalent to

$$R_{2n+1} = \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{1}{R_{2n-1} - d_{2n-1}}. \tag{4}$$

We can check (4) directly for $n = 0$. Assume $n \ge 1$. Then the right-hand side of (4) is equal to

$$\begin{aligned}\frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{2x}{1-3x-\sqrt{(1-x)(1-5x)}} &= \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{2x\left(1-3x+\sqrt{(1-x)(1-5x)}\right)}{(1-3x)^2 - (1-6x+5x^2)}\\ &= \frac{1}{x} - \frac{1}{d_{2n-1}} - \frac{1-3x+\sqrt{(1-x)(1-5x)}}{2x}\\ &= 3 - \frac{1}{d_{2n-1}} + \frac{1-3x-\sqrt{(1-x)(1-5x)}}{2x}.\end{aligned}$$

Since $3 - 1/d_{2n-1} = d_{2n+1}$ by Lemma 2.2, we are done. $\square$

Proof of Theorem 2.4. It follows from Lemma 2.5 that

$$\frac{3}{2}-\frac{1}{2}\sqrt{\frac{1-5x}{1-x}} = R_{-1} = \frac{1}{1-xR_0} = \cfrac{1}{1-\cfrac{d_0x}{1-xR_1}} = \cfrac{1}{1-\cfrac{d_0x}{1-\cfrac{d_1x}{1-xR_2}}} = \cdots .$$

Continuing, and taking a limit, gives the S-fraction for R_{-1}. □

By (1), Theorem 2.4 and Corollary 2.3, we obtain the following which proves Theorem 1.1.

$$\sum_{n\ge 0} \#\,\mathrm{NC}_2(n)x^n = \frac{3}{2}-\frac{1}{2}\sqrt{\frac{1-5x}{1-x}} = \sum_{n\ge 0}\mathrm{Dyck}_n(d)x^n = \sum_{n\ge 0}\mathrm{Mot}_n(b,\lambda)x^n$$

3. A combinatorial proof

Let b, λ and d be the sequences defined in (2) and (3).

Recall that in the previous section we have shown that $\mathrm{Dyck}_n(d) = \mathrm{Mot}_n(b,\lambda)$ by changing a Dyck path of length $2n$ to a Motzkin path of length n. We can do the same thing after deleting the first and the last steps of a Dyck path. More precisely, for a Dyck path of length $2n$, we delete the first and the last steps, take two steps at a time in the remaining $2n-2$ steps, and change UU, UD, DU and DD, respectively, to U, H, H and D. Then we obtain a Motzkin path of length $n-1$. This argument shows that

$$\mathrm{Dyck}_n(d) = d_0\cdot \mathrm{Mot}_{n-1}(\alpha,\beta) = \mathrm{Mot}_{n-1}(\alpha,\beta), \tag{5}$$

where $\alpha_n = d_{2n}+d_{2n+1}$ and $\beta_n = d_{2n+1}d_{2n+2}$. By (3) and Lemma 2.2, we have $\alpha = (2,3,3,\ldots)$ and $\beta = (1,1,\ldots)$. Note that we can also prove Theorem 2.4 using (5).

To find a connection between $\mathrm{Mot}_{n-1}(\alpha,\beta)$ and $\mathrm{NC}_2(n)$ we need the following definition.

A *Schröder path* of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$ and double horizontal steps $H^2 = HH = (2,0)$ that never goes below the x-axis. Let $\mathrm{SCH}_{\mathrm{even}}(n)$ denote the set of Schröder paths of length $2n$ such that all horizontal steps have even height.

Proposition 3.1. *Let $\alpha = (2,3,3,\ldots)$ and $\beta = (1,1,\ldots)$. Then, for $n\ge 1$, we have*

$$\mathrm{Mot}_n(\alpha,\beta) = \#\,\mathrm{SCH}_{\mathrm{even}}(n).$$

Proof. From a Motzkin path of length n we obtain a Schröder path of length $2n$ as follows. Change U and D to UU and DD respectively. For a horizontal step H, if its height is 0, we change it to either UD or HH, and if its height is greater than 0, we change it to either UD, DU or HH. Then we get an element of $\mathrm{SCH}_{\mathrm{even}}(n)$. Since the weight of a horizontal step H in the Motzkin path is equal to the number of choices, the theorem follows. □

Remark 1. The definition of $\mathrm{SCH}_{\mathrm{even}}(n)$ in [2] is the set of Schröder paths of length $2n$ which have no peaks at even height. From such a path, by changing all the horizontal steps at odd height to peaks, we get a Schröder path whose horizontal steps are all at even height, and this transformation is easily seen to be a bijection.

Kim [2] found a bijection between $\mathrm{NC}_2(n)$ and $\mathrm{SCH}_{\mathrm{even}}(n-1)$. Using Kim's bijection in [2], Proposition 3.1, (5) and Corollary 2.3 we finally get the following sequence of identities which implies Theorem 1.1:

$$\#\,\mathrm{NC}_2(n) = \#\,\mathrm{SCH}_{\mathrm{even}}(n-1) = \mathrm{Mot}_{n-1}(\alpha,\beta) = \mathrm{Dyck}_n(d) = \mathrm{Mot}_n(b,\lambda).$$

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